Journal of Statistical Physics, Vol. 56, Nos. 1/2, 1989

Global Existence in L^1 for the Modified Nonlinear Enskog Equation in $\mathbb{R}^3$

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Received September 6, 1988; final January 24, 1989

A global existence theorem with large initial data in L^1 is given for the modified Enskog equation in $\mathbb{R}^3$. The method, which is based on the existence of a Liapunov functional (analog of the H-Boltzmann theorem), utilizes a weak compactness argument in L^1 in a similar way to the DiPerna–Lions proof for the Boltzmann equation. The existence theorem is obtained under certain condition on the behavior of the geometric factor Y. The condition on Y amounts to the fact that the L^1 norm of the collision term grows linearly when the local density tends to infinity.

KEY WORDS: Enskog equation; Boltzmann equation; kinetic theory.

1. INTRODUCTION

In this paper I outline the proof of a global existence theorem for the modified Enskog equation. The proof utilizes ideas exploited recently by DiPerna and Lions to study existence of solutions to the Boltzmann equation. The DiPerna–Lions work[(10)] is based on a new averaging lemma for first-order hyperbolic differential operators by Golse *et al.*[(11)] and has had a large impact on rigorous kinetic theory. Because of the importance of these ideas, and because parallel arguments are to be used in the current work on the Enskog equation, I present in the next section a careful summary of this *averaging method* applied to the Boltzmann equation, and brief comments on its value and its weaknesses. In Section 3, I return to the Enskog equation, and derive new *a priori* estimations for the modified equation. Finally, in Section 4 I indicate that these estimations enable the existence theory to be analyzed in the context of the averaging method.

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0022-4715/89/0700-0159$06.00/0

The Enskog equation, derived in 1921 by Enskog[12] to take account explicitly of the finite diameter of molecules, is a successful kinetic model of a dense gas consisting of hard spheres. A modified Enskog equation can be derived from the BBGKY hierarchy by computing the reduced N-particle distribution function from a special grand canonical formalism and arriving at a closure relation for the two-particle distribution function (see Resibois[1,2] and the bibliography in ref. 3). The result is

$$\frac{\partial f}{\partial f} + v \frac{\partial f}{\partial x} = E(f) \tag{1.1}$$

where $f(t, x, v)$ is the one-particle distribution function with $t \geqslant 0$, $x, v \in \mathbb{R}^3$, and the collision operator $E(f)$ defined by $E(f) = E^+(f) - E^-(f)$ with

$$\begin{aligned} E^+(t) = a^2 \iint_{R^3 \times S^2_+} & Y(n(t, x), n(t, x - a\varepsilon))\, f(t, x, v') \\ & \times f(t, x - a\varepsilon, w') \langle \varepsilon, v - \omega \rangle \, d\varepsilon \, dw \end{aligned} \tag{1.2a}$$

$$\begin{aligned} E^-(f) = a^2 \iint_{R^3 \times S^2_+} & Y(n(t, x), n(t, x + a\varepsilon))\, f(t, x, v) \\ & \times f(t, x + a\varepsilon, w) \langle \varepsilon, v - w \rangle \, d\varepsilon \, dw \end{aligned} \tag{1.2b}$$

Here, a denotes the hard sphere diameter, $\langle \cdot, \cdot \rangle$ is the inner product in R^3, $\varepsilon \in S^2_+ = \{\varepsilon \in R^3 \colon |\varepsilon| = 1, \ \langle v - w, \varepsilon \rangle \geqslant 0\}$, and the velocities after the collision, v', w', are given by

$$v' = v - \varepsilon \langle \varepsilon, v - w \rangle, \qquad w' = w + \varepsilon \langle \varepsilon, v - w \rangle \tag{1.3}$$

The function $n(t, x) = \int_{R^3} f(t, x, v)\, dv$ is the local density of the gas, and the geometric factor Y is a function of the local density at x and $x \pm a\varepsilon$. One notes that in the original Enskog equation Y depends only on the density at the point of contact, i.e., at the point $x \pm \frac{1}{2} a\varepsilon$.

An essential differences between the modified equation and the original equation is that for the modified equation there is an analog of the Boltzmann H-theorem. Indeed, Resibois showed[2] that $H(t)$ given by

$$H(t) = \sum_{N=0}^{\infty} \int d\Gamma^N \rho_N(t) \log[N! \, \rho_N(t)] \tag{1.4}$$

is nonincreasing in $t \geqslant 0$, where $\rho_N(t)$ is the approximate N-particle distribution function,[2] and that (at least formally) the modified Enskog equation drives the gas confined in a box with periodic boundary conditions toward the absolute Maxwellian.

The function $H(t)$ given in (1.4) can be rewritten in the form (ref. 2, p. 600)

$$H(t) = \iint f(t, x, v) \log f(t, x, v)\, dv\, dx + H^v(t) \tag{1.5}$$

where $f(t, x, v)$ is the solution to (1.1), and the potential part $H^v(t)$ is given in terms of Resibois' grand canonical formalism, but, unfortunately, not explicitly in terms of $f(t, x, v)$ and Y. I show in Section 2 that this potential difficulty in utilizing the H-function can be overcome.

I end this section with a brief review of known existence theorems for the original or modified Enskog equation (see ref. 3 for a more detailed review). The first local in time existence theorem was obtained by Lachowicz.(4) A global in time existence theorem was obtained by Toscani and Bellomo(5) in the case of a perturbation of the vacuum. I showed(6) that the solution obtained in ref. 5 is actually a classical solution to (1.1) if the initial datum is smooth. Furthermore, the asymptotic behavior of solutions was obtained in ref. 6. All of the above results deal with the original Enskog equation, but with easy modifications can be extended to the modified Enskog equation.

The quoted results fall in either of two categories: small initial data or local in time existence results. For large initial data, Cercignani(7) obtained global in time L^1 solutions in the case of one space dimension and $Y \equiv 1$. Arkeryd(8) extended Cercignani's result to two space dimensions using a weak compactness argument in L^1, however, with the range of integration with respect to ε extended to the whole sphere S^2, together with the assumption that $Y \equiv 1$. It is worth noting that this alteration in the range of integration has a significant effect on the dynamics of the Enskog equation. In fact, the original Enskog equation and the modified equation, with integration over S^2_+, distinguish between forward and backward (time-reversed) collisions, while the Boltzmann equation and the alteration above, with integration over S^2, are symmetric under forward and backward collisions. Recently, Arkeryd(9) has obtained global existence for $Y \equiv 1$ and the assumption of bounded velocities. More precisely, he replaced $\langle \varepsilon, v - w \rangle$ in (1.2) by $\langle \varepsilon, v - w \rangle W_j$, where $W_j = 1$ if $v^2 + w^2 \leqslant 4^j$ and $W_j = 0$ otherwise.

In this paper, the proof of a global existence theorem is based on the fact that one can find a Liapunov functional to (1.1) if $Y(\cdot, \cdot)$ is a symmetric function of two variables. To utilize the Liapunov functional further, we need an additional assumption on the behavior of Y when $n(t, x)$ tends to infinity [see (3.9)]. This condition implies that the L^1 norm of the collision term $E(f)$ grows linearly when $n(t, x)$ tends to infinity. We want to point out that in one space dimension condition (3.9) is superfluous, and the only additional assumption that is needed (except, of course, for the symmetry) is the boundedness of Y. This observation generalizes Cercignani's result[7]; indeed, he assumed that $Y \equiv 1$, which is symmetric and bounded.

2. SKETCH OF THE DiPERNA–LIONS RESULT

Consider the Cauchy problem for the Boltzmann equation

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = Q(f), \qquad f(0, x, v) = f_0(x, v) \tag{2.1}$$

where $x \in R^3$, $v \in R^3$, and $Q(f)$ is the Boltzmann collision operator. $Q(f)$ can be written in the form $Q(f) = Q^+(f) - Q^-(f)$, where

$$Q^+(f) = \iint_{R^3 \times S_+^2} f(t, x, v')\, f(t, x, w')\, B(\theta, v-w)\, d\varepsilon\, dw$$

$$Q^-(f) = \iint_{R^3 \times S_+^2} f(t, x, v)\, f(t, x, w)\, B(\theta, v-w)\, d\varepsilon\, dw$$

Here v' and w' are given by Eq. (1.3). The angle $\theta \in [0, \pi/2]$ is defined by $\cos\theta = \langle v-w, \varepsilon\rangle / |v-w|$, and $B(\theta, v-w)$ is the scattering kernel with the usual angular cutoff. For inverse power potentials, $\mathscr{F}(r) = r^{-s}$, $B(\theta, v-w) = b(\theta)\, |v-w|^{(s-4)/s}$ with $s > 2$. For the hard spheres model, $B(\theta, v-w) = |v-w| \cos\theta = \langle v-w, \varepsilon\rangle$. The DiPerna–Lions result to be outlined here will cover all soft and hard potentials; in fact, it will work for all $B(\theta, v-w) = b(\theta)\, |v-w|^{\lambda}$ with $-3 < \lambda < 2$ and $\int_{S_+^2} b(\theta)\, d\varepsilon < \infty$. We say that f is a mild solution to (2.1) if $Q^{\pm}(f)(t, x, v) \in L^1(0, T)$ a.e. in $(v, x) \in R^3 \times R^3$ and

$$f^{\#}(t, x, v) - f^{\#}(s, x, v) = \int_s^t Q(f)^{\#}\, (\tau, x, v)\, d\tau$$

for any $0 < s < t \leqslant T$ with $f^{\#}(t, x, v) = f(t, x + tv, v)$.

The Boltzmann collision operator Q possesses the following two fundamental properties. For $\psi \in C(R^3 \times R^3)$ and $f \in C_0(R^3 \times R^3)$ we have

$$\int_{R^3} \psi Q(f)\, dv = \iiint_{R^3 \times R^3 \times S_+^2} [\psi(x, v) + \psi(x, w) - \psi(x, v') - \psi(x, w')] \times [f(x, v')\, f(x, w') - f(x, v)\, f(x, w)]\, B(\theta, v-w)\, d\varepsilon\, dw\, dv \tag{2.2}$$

and for $0 \leqslant f \in C_0(R^3 \times R^3)$ we have

$$\int_{R^3} Q(f) \log f \, dv \leqslant 0 \tag{2.3}$$

As a result of (2.2) and (2.3) we may conclude that if $f(t, x, v)$ is a smooth and nonnegative solution to (2.1) with a nonnegative initial value f_0 satisfying

$$\iint_{R^3 \times R^3} (1 + v^2 + x^2 + |\log f_0|) f_0 \, dv \, dx = C_0 < \infty \tag{2.4}$$

then

$$\sup_{0 \leqslant t \leqslant T} \iint_{R^3 \times R^3} (1 + v^2 + x^2 + |\log f|) f \, dv \, dx \leqslant C_T \tag{2.5}$$

where C_T depends only on T and C_0.

Now, let Q_n denote a suitable approximation of Q (defined later) for which (2.2) and (2.3) are satisfied and such that the initial value problem

$$\frac{\partial f_n}{\partial t} + v \frac{\partial f_n}{\partial x} = Q_n(f_n), \qquad f_n(0, x, v) = f_0(x, v) \tag{2.6}$$

has a nonnegative solution on $[0, T]$, $T > 0$. Then (2.5) and the Dunford–Pettis theorem[13] imply that $\{f_n\}$ is relatively weakly compact in $L^1((0, T) \times R^3 \times R^3)$. Without loss of generality we may assume that $f_n \to f$ weakly in L^1, where $0 \leqslant f \in L^1$. The idea, of course, is to find Q_n for which (2.6) can be solved for each $n \geqslant 1$ and such that f satisfies the Boltzmann equation (2.1) in some specified sense.

One should point out that the solution of a sequence of approximate initial value problems and the convergence of the approximate solutions is a stage which has been reached by many authors. Arkeryd[14] solved (2.6) for some truncated version of Q. However, he was unable to show that the weak limit f of a sequence f_n satisfies the Boltzmann equation. The authors in refs. 15 and 16 replaced $v\, \partial/\partial x$ by its finite difference approximation and also solved the simplified problem. As in ref. 14, they were unable to pas with the limit in the Eq. (2.1). A stumbling block in these attempts was that Q is not weakly continuous in $L^1(R^3 \times R^3)$, and, in fact, is even difficult to define in a reasonable way in $L^1(R^3 \times R^3)$. However, Arkeryd[17] did use a weak compactness argument for the space-independent problem. In this case the collision operator is weakly sequentially continuous, i.e., $Q(f_n) \to Q(f)$ weakly in $L^1(R^3)$ if $f_n \to f$ weakly in $L^1(R^3)$, and therefore f satisfies the homogeneous Boltzmann equation.

The success of DiPerna and Lions relies on two new arguments applied to the Boltzmann equation. The first deals with the notion of a so-called renormalized solution. We say that $f \in C([0, T] \times L^1_+(R^3 \times R^3))$ is a renormalized solution to (2.1) if

$$\frac{1}{1+f} Q^{\pm}(f) \in L^1((0, T) \times B_R \times B_R)$$

for any $R > 0$ and

$$\frac{\partial}{\partial t} \log(1+f) + v \frac{\partial}{\partial x} \log(1+f) = \frac{1}{1+f} Q(f) \tag{2.7}$$

in $\mathscr{D}'((0, \infty) \times R^3 \times R^3)$. DiPerna and Lions showed that f is a renormalized solution to (2.1) if and only if it is a mild solution and $Q^{\pm}(f)/(1+f) \in L^1_{\text{loc}}$.

The second concept is a new compactness argument due to Golse *et al.*,[11] which applies to general transport equations. Suppose that $f_n \in L^1(0, T) \times R^3 \times R^3)$ and $g_n \in L^1_{\text{loc}}((0, T) \times R^3 \times R^3)$ satisfy

$$T_v f_n \overset{\text{def}}{=} \frac{\partial f_n}{\partial t} + v \frac{\partial f_n}{\partial x} = g_n \tag{2.8}$$

in $\mathscr{D}'((0, T) \times R^3 \times R^3)$. If we know that for each compact set K of $(0, T) \times R^3 \times R^3$ the sequences $\{f_n\}$ and $\{g_n\}$ are relatively weakly compact in $L^1((0, T) \times R^3 \times R^3)$ and $L^1(K)$, respectively, then the averaging lemma asserts that for all $\varphi \in L^\infty((0, T) \times R^3 \times R^3)$ the set $\{\int_{R^3} \varphi f_n \, dv\} = \{\int_{R^3} \varphi T_v^{-1} g_n \, dv\}$ is relatively compact in $L^1((0, T) \times R^3)$. In other words, the velocity-averaged operator T_v^{-1} behaves in a similar way to the inverse of an elliptic operator. We recall that T_v^{-1} may be singular only on the set of the characteristic direction. Velocity averaging compensates for the lack of regularity in the characteristic direction of the hyperbolic operator.

In view of (2.7), in order to utilize the above compactness argument we need weak compactness of $\{Q(f_n)/(1+f_n)\}$ in $L^1((0, T) \times R^3 \times B_R)$ for all $R > 0$, where $B_R = \{v \in R^3 \colon |v| \leqslant R\}$. Since the weak compactness of $\{Q^- f_n/(1+f_n)\}$ follows in a relatively simple way, we need to show the weak compactness of $\{Q^+ f_n/(1+f_n)\}$. This is possible due to the following new estimation of the gain term Q^+ provided by DiPerna and Lions:

$$Q^+ f_n \leqslant M Q^- f_n + \frac{1}{\log M} e(f_n) \tag{2.9}$$

for all $M > 1$, where

$$e(f_n) = \iint_{R^3 \times S^2_+} [f_n(t, x, v') f_n(t, x, w') - f_n(t, x, v) f_n(t, x, w)] \times B(\theta, v - w) \log \frac{f_n(t, x, v') f_n(t, x, w')}{f_n(t, x, v) f_n(t, v, w)} d\varepsilon\, dw$$

In view of (2.2) and (2.3), we have $e(f_n) \geqslant 0$, and since we also have the entropy identity

$$\frac{d}{dt} \iint_{R^3 \times R^3} f_n \log f_n \, dv\, dx + \frac{1}{4} \iint_{R^3 \times R^3} e(f_n)\, dv\, dx = 0 \tag{2.10}$$

we see that $\{e(f_n)\}$ is bounded in $L^1((0, T) \times R^3 \times R^3)$ uniformly in $n \geqslant 1$.

We are now in a position to summarize the main points of the DiPerna–Lions proof. We approximate Q by its truncated version

$$Q_n f_n = \frac{1}{1 + (1/n) \int_{R^3} f_n \, dv} \iint_{R^3 \times S^2_+} [f_n(x, v') f_n(x, w') - f_n(x, v) f_n(x, w)]\, B_n(\theta, v - w)\, d\varepsilon\, dw$$

where

$$B_n(\theta, v - w) = [\cos^2 \theta (1/n + \cos^2 \theta)^{-1}] \times \inf\{1, |v - w|^{1/n}\} \times B(\theta, v - w)$$

if $v^2 + w^2 \leqslant n$, and 0 otherwise. For such Q_n the problem (2.6) can be solved uniquely on $[0, T]$ for any $T > 0$. Indeed, Q_n is Lipschitz in $L^1((0, T) \times R^3 \times R^3)$ with Lipschitz constant depending only on n. Furthermore, f_n is a smooth, nonnegative solution to (2.6). As before, (2.2) and (2.3) imply that, by passing to a subsequence if necessary, $f_n \to f$ weakly in $L^1((0, T) \times R^3 \times R^3)$ and, by the averaging lemma,

$$\int_{R^3} \varphi f_n \, dv \to \int_{R^3} \varphi f \, dv \qquad \text{strongly in} \quad L^1((0, T) \times R^3) \tag{2.11}$$

for all $\varphi \in L^\infty((0, T) \times R^3 \times R^3)$.

Next, using (2.11), one may show that for each $\varphi \in L^\infty((0, T) \times R^3 \times R^3)$ with compact support,

$$\int_{R^3} Q(f_n) \varphi \, dv \to \int_{R^3} Q(f) \varphi \, dv \tag{2.12}$$

in measure on $(0, T) \times B_R$. Properties (2.11) and (2.12) are enough to show that f is a mild solution to (2.1).

The procedure for solving (2.1) may be analyzed as follows. The Boltzmann equation has naturally built into its structure the weak compactness argument [properties (2.2) and (2.3)]. The question is whether the weak limit of a sequence of solutions is again a solution. In spite of the fact that Q is not weakly continuous in $L^1(R^3 \times R^3)$, the answer to this question is affirmative. Indeed, the weak limit of a sequence of classical solutions is a renormalized, or equivalently, mild solution to (2.1). In other words, the set of renormalized solutions is closed in the weak topology of $L^1((0, T) \times R^3 \times R^3)$.

The DiPerna–Lions paper has had great impact in part because the methods are applicable to a variety of problems in kinetic theory, and also because it answers (affirmatively) for the first time the question of whether the full nonlinear Boltzmann equation has solutions, valid for all time, when the initial value is far from the equilibrium solution (the Maxwellian). This has been an open problem not just from a rigorous point of view. Since derivations of the Boltzmann equation lean—at least implicity—on the assumption that the gas is not too far from its equilibrium configuration, there has not been even good physical intuition as to whether global solutions should exist for initial configurations arbitrarily far from equilibrium.

On the other hand, weak compactness arguments necessarily suffer from a substantial drawback. No information is offered as to the uniqueness of the derived solution. And, indeed, to date, no ideas have been offered to decide if the sequence $\{f_n\}$ obtained by DiPerna and Lions has a unique limit point, and, if so, if the limit is the unique solution of the Boltzmann equation.

3. BASIC IDENTITIES AND *A PRIORI* ESTIMATIONS FOR THE MODIFIED ENSKOG EQUATION

In order to apply averaging techniques to the study of the modified Enskog equation, it is first necessary to obtain a number of *a priori* estimates for its solution.

First, let us note that the factor $Y(n(t, x), n(t, x - a\varepsilon))$, which in general is a functional of the local density at x and $x - a\varepsilon$, arises from the Resibois' formalism as a symmetric functional of $n(t, x)$ and $n(t, x - a\varepsilon)$. In fact, this property is crucial in the derivation of the H-theorem, and is necessary, in any case, to obtain conservation laws. We shall assume this symmetry throughout, namely, $Y(\tau, \sigma) = Y(\sigma, \tau)$, $\sigma, \tau \geqslant 0$. The first property of $E(f)$ we indicate is an analog of corresponding identities for the Boltzmann collision operator. For $\psi \in C^\infty(R^3 \times R^3)$ and $f \in C_0(R^3 \times R^3)$,

$$\iint_{R^3 \times R^3} \psi(x, v)\, E(f)\, dv\, dx$$
$$= \frac{a^2}{2} \iiiint_{R^3 \times R^3 \times R^3 \times S_+^2} [\psi(x, v') + \psi(x + a\varepsilon, w')$$
$$- \psi(x, v) - \psi(x + a\varepsilon, w)] \times f(x, v)\, f(x + a\varepsilon, w)\, Y(n(x),$$
$$n(x + a\varepsilon)) \langle \varepsilon, v - w \rangle\, d\varepsilon\, dw\, dv\, dx \tag{3.1}$$

While Resibois must have been aware of this identity, it was never stated explicitly [see, however, the identities (35) and (37) in ref. 2, where ψ is replaced by $\log f(x, v)$].

For f a nonnegative solution to (1.1), and ignoring at this stage any integrability conditions, we define

$$\Gamma(t) = \iint_{R^3 \times R^3} f(t, x, v) \log f(t, x, v)\, dv\, dx - \int_0^t I(s)\, ds \tag{3.2}$$

where

$$I(t) = \frac{1}{2} a^2 \iiiint_{R^3 \times R^3 \times R^3 \times S_+^2} [f(t, x - a\varepsilon, w)\, Y(n(t, x), n(t, x - a\varepsilon))$$
$$- f(t, x + a\varepsilon, w)\, Y(n(t, x), n(t, x + a\varepsilon))]$$
$$\times f(t, x, v) \langle \varepsilon, v - w \rangle\, d\varepsilon\, dw\, dv\, dx$$

Now, multiplying (1.1) by $1 + \log f$ and integrating over $(x, v) \in R^3 \times R^3$, we have

$$\frac{d\Gamma}{dt} = \iint_{R^3 \times R^3} E(f) \log f\, dv\, dx - I(t) \tag{3.3}$$

Using the identities (35) and (37) of ref. 2 together with the inequality $y(\log y - \log z) \geqslant y - z$ for $y, z > 0$, we obtain

$$\frac{d\Gamma}{dt} \leqslant 0 \tag{3.4}$$

The inequality (3.4) shows that $\Gamma(t)$ is a Liapunov functional for (1.1). $\Gamma(t)$ displays the irreversibility of the system governed by the modified Enskog equation. It also can be considered as the analog of the H-function for (1.1). Note that Γ is defined explicitly in terms of $f(t, x, v)$ and Y, in contrast to the H-function (1.4) [or (1.5)] obtained by Resibois.[2] Furthermore, in the dilute gas limit, when the modified Enskog equation becomes the Boltzmann equation, the function $\Gamma(t)$ reduces to the Boltzmann H-function.

In order to obtain *a priori* estimations on the solution $f(t, x, v)$ to (1.1), let us assume throughout this section that f is a smooth nonnegative solution with initial value $f_0(x, v)$ satisfying

$$\iint_{R^3 \times R^3} (1 + v^2 + x^2 + |\log f_0|) f_0 \, dv \, dx \leqslant C_0 < \infty \tag{3.5}$$

Our first *a priori* estimations are the following conservation laws, which follow from (3.1) with $\psi = 1, v, v^2$:

$$\iint_{R^3 \times R^3} f(t, x, v) \, dv \, dx = \iint_{R^3 \times R^3} f_0(x, v) \, dv \, dx \tag{3.6}$$

$$\iint_{R^3 \times R^3} v f(t, x, v) \, dv \, dx = \iint_{R^3 \times R^3} v f_0(x, v) \, dv \, dx \tag{3.7}$$

$$\iint_{R^3 \times R^3} v^2 f(t, x, v) \, dv \, dx = \iint_{R^3 \times R^3} v^2 f_0(x, v) \, dv \, dx \tag{3.8}$$

So far the only property of Y employed has been the symmetry $Y(\tau, \sigma) = Y(\sigma, \tau)$, $\tau, \sigma \geqslant 0$.

In order to utilize (3.2) and (3.4) further, we need an additional assumption on Y:

$$\sup_{\tau, \sigma \geqslant 0} \tau Y(\tau, \sigma) = M_Y < \infty \tag{3.9}$$

Now it is an easy exercise to show that (3.9) together with the symmetry of Y imply

$$\sup_{0 \leqslant t \leqslant T} |I(t)| \leqslant 4\pi a^2 M_Y \iint_{R^3 \times R^3} (1 + v^2) f_0(x, v) \, dv \, dx \tag{3.10}$$

In view of (3.10), one immediately sees from (3.2) and (3.4) that

$$\begin{aligned} \sup_{0 \leqslant t \leqslant T} & \iint_{R^3 \times R^3} f \log f \, dv \, dx \\ & \leqslant \iint_{R^3 \times R^3} f_0 \log f_0 \, dv \, dx + (1 + T) 4\pi a^2 M_Y \\ & \quad \times \iint_{R^3 \times R^3} (1 + v^2) f_0(x, v) \, dv \, dx \end{aligned} \tag{3.11}$$

We further notice that in the case of one space dimension one can obtain estimation (3.10), and hence (3.11), without the assumption (3.9). Indeed, using the same technique as in the lemma of ref. 7, p. 216, one obtains

$$\sup_{0\leqslant t\leqslant T} |I(t)| \leqslant 16\pi a[\sup_{\tau,\sigma\geqslant 0} Y(\tau,\sigma)]\left[\iint (1+v^2) f_0(x,v)\, dv\, dx\right]^2 = C_1 \tag{3.12}$$

As before, (3.12) implies

$$\sup_{0\leqslant t\leqslant T} \iint_{R^3\times R^3} f\log f\, dv\, dx \leqslant \iint_{R^3\times R^3} f_0 \log f_0\, dv\, dx + (1+T) C_1 \tag{3.13}$$

An important physical case when (3.9) holds is provided by Y with compact support, support $\{Y\}\subset [0, n_c]\times[0, n_c]$. Since the diameter of the hard spheres is greater than zero, n_c can be interpreted as "the density at which the spheres become strictly packed" (see ref. 7, p. 214).

Another estimation can be obtained by multiplying (1.1) by $(x-tv)^2$, integrating by parts over $x\in R^3$, and using (3.1) along with the equality

$$(x-tv')^2 + (x+a\varepsilon - tw')^2 = (x-tv)^2 + (x+a\varepsilon - tw)^2 - 2at\langle \varepsilon, v-w\rangle$$

for $x, v, w\in R^3$, $t\in R$, $a>0$, $\varepsilon\in S^2_+$, and v', w' given in (1.3). The result is

$$\begin{aligned}\frac{d}{dt}&\iint_{R^3\times R^3} (x-tv)^2 f(t,x,v)\, dv\, dx\\ &= -a^3 t \iiiint_{R^3\times R^3\times R^3\times R^3} \langle \varepsilon, v-w\rangle^2\, Y(n(t,x), n(t, x+a\varepsilon))\\ &\quad \times f(t,x,v)\, f(t, x+a\varepsilon, w)\, d\varepsilon\, dw\, dv\, dx\end{aligned} \tag{3.14}$$

In view of (3.9), the right-hand side of (3.14) is bounded by a constant depending only on f_0. Therefore (3.14) implies

$$\sup_{0\leqslant t\leqslant T} \iint_{R^3\times R^3} x^2 f(t,x,v)\, dv\, dx \leqslant C_2 \tag{3.15}$$

where C_2 depends on T, $\iint_{R^3\times R^3} x^2 f_0\, dv\, dx$, and $\iint_{R^3\times R^3} (1+v^2) f_0\, dv\, dx$.

Combining all the above, we have for initial data satisfying (3.5),

$$\sup_{0\leqslant t\leqslant T} \iint_{R^3\times R^3} (1+v^2+x^2+|\log f|) f\, dv\, dx \leqslant C_T \tag{3.16}$$

where C_T depends on T and f_0.

Our final estimation deals with the analog of the gain–loss estimation (2.9) for the Boltzmann equation. First, we have, for each $M > 1$,

$$E^+(f) \leqslant Ma^2 \iint_{R^3 \times S_+^2} Y(n(f, x), n(t, x - a\varepsilon)) \times f(t, x, v)\, f(t, x - a\varepsilon, w) \langle \varepsilon, v - w \rangle \, d\varepsilon \, dw + \frac{1}{\log M} \alpha(f) \tag{3.17}$$

where

$$\alpha(f) = a^2 \iint_{R^3 \times S_+^2} Y(n(t, x), n(t, x - a\varepsilon))\, f(t, x, v')\, f(t, x - a\varepsilon, w') \times \left| \log \frac{f(t, x, v')\, f(t, x - a\varepsilon, w')}{f(t, x, v)\, f(t, x - a\varepsilon, w)} \right| \langle \varepsilon, v - w \rangle \, d\varepsilon \, dw$$

Also, by using (35) and (37) of ref. 2, we obtain

$$\iint_{R^3 \times R^3} f(t, x, v) \log f(t, x, v) \, dv \, dx - \iint_{R^3 \times R^3} f_0 \log f_0 \, dv \, dx = \int_0^t \iiiint_{R^3 \times R^3 \times R^3 \times S_+^2} h(f) \, d\varepsilon \, dw \, dv \, dx \, ds \tag{3.18}$$

for

$$h(f) = -\frac{a^2}{2} Y(n(s, x), n(s, x - a\varepsilon))\, f(s, x, v')\, f(s, x - a\varepsilon, w') \times \langle \varepsilon, v - w \rangle \log \frac{f(s, x, v')\, f(s, x - a\varepsilon, w')}{f(s, x, v)\, f(s, x - a\varepsilon, w)}$$

The inequality $z(\log z - \log y) \geqslant z - y$, together with the assumption (3.9) on Y, implies that for $h^+(f) = \max\{h(f), 0\}$ we have

$$\int_0^T \iiiint_{R^3 \times R^3 \times R^3 \times S_+^2} h^+(f) \, d\varepsilon \, dw \, dv \, dx \, ds \leqslant \text{const} \cdot \iint_{R^3 \times R^3} (1 + v^2) f_0 \, dv \, dx$$

Next, because of (3.5) and (3.16), the left-hand side of (3.18) is bounded. Hence, for $h^-(f) = \max\{-h(f), 0\}$ we have

$$\int_0^T \iiiint_{R^3 \times R^3 \times R^3 \times S_+^2} h^-(f) \, d\varepsilon \, dw \, dv \, dx \, ds \leqslant \text{const}(C_0, C_T)$$

Finally, since

$$\int_0^T \iint_{R^3 \times R^3} \alpha(f)\, dv\, dx\, ds$$

$$= 2 \int_0^T \iiiint_{R^3 \times R^3 \times R^3 \times S_+^2} [h^+(f) + h^-(f)]\, d\varepsilon\, dw\, dx\, dx\, ds$$

we may obtain a bound on the norm of $\alpha(f)$ in $L^1(0, T) \times R^3 \times R^3)$ that depends only on C_0 and C_T.

4. EXISTENCE THEOREM

The results of the previous section, in particular estimations (3.16) and (3.17), place the modified Enskog equation in the framework of the DiPerna–Lions method developed for the Boltzmann equation.[10]

We have the following result.

Theorem. Suppose that Y satisfies (3.9) and $f_0 \geqslant 0$ satisfies (2.4). Then there exists a mild solution to (1.1).

The idea of the proof is to approximate (1.1) by considering

$$\frac{\partial f^k}{\partial t} + \frac{\partial f^k}{\partial x} = E_k(f^k) \tag{4.1}$$

where $E_k(f)$ is the modified Enskog operator with $\langle \varepsilon, v - w \rangle$ replaced by $\langle \varepsilon, v - w \rangle \times [\cos^2 \theta (1/k + \cos^2 \theta)^{-1}] \times W_k$ and $Y(\tau, \sigma)$ replaced by

$$Y_k(\tau, \sigma) = [1 + (1/k)\tau]^{-1} [1 + (1/k)\sigma]^{-1} Y(\tau, \sigma)$$

Here $W_k = 1$ for $v^2 + w^2 \leqslant k$ and $W_k = 0$ otherwise, and $\cos \theta = \langle v - w, \varepsilon \rangle / |v - w|$. Next, approximating f_0 by convolution to obtain $\tilde{f}_0 \in C_0^\infty(R^3 \times R^3)$ and $\tilde{f}_0 \geqslant 0$, we show that f^k is smooth. By (3.16), one may show that $\{f^k\}$ is weakly relatively compact in $L^1((0, T) \times R^3 \times R^3)$. Hence, by the averaging lemma of Golse *et al.*,[11] we obtain that for all $\varphi \in L^\infty((0, T) \times R^3 \times R^3)$, $\{\int_{R^3} f^k(t, x, v)\, \varphi(t, x, v)\, dv\}_{k=1}^\infty$ is relatively compact in $L^1((0, T) \times R^3)$. This means that after passing to a subsequence, if necessary, we have

$$\int_{R^3} f^k(t, x, v)\, dv \xrightarrow[k \to \infty]{} \int_{R^3} f(t, x, v)\, dv \quad \text{a.e. in} \quad (t, x) \in [0, T] \times R^3$$

Therefore

$$Y_k\left(\int_{R^3} f^k(t, x, v)\,dv, \int_{R^3} f^k(t, x \pm a\varepsilon, v)\,dv\right)$$

$$\xrightarrow[k \to \infty]{} Y\left(\int_{R^3} f(t, x, v)\,dv, \int_{R^3} f(t, x \pm a\varepsilon, v)\,dv\right)$$

a.e. in $(t, x, \varepsilon) \in [0, T] \times R^3 \times S^2_+$. From this the DiPerna–Lions line of argument can be used to prove our theorem. Detailed proofs will be presented in a separate paper in preparation.

In closing, I note that the results outlined here can easily be extended to the modified Enskog equation in bounded spatial domains with periodic boundary conditions. In this case the explicit x^2 term in the estimations (3.5) and (3.16) is unnecessary.

ACKNOWLEDGMENTS

This paper has benefited greatly from the comments of William Greenberg. This work was supported in part by DOE grant DEF60587ER25033 and NSF grant DM587015050.

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